M 317 Final Exam

Explain each step of your solution to the following problems as completely as you possibly can. If you are making use of some theorem, say so and explain why you can use the theorem. If you assume something without justifying it, I will assume you do not know how to explain it.

1. Suppose f and g are continuous on R and that $f(x) = g(x)$ for every rational number x. Prove that $f(x) = g(x)$ for every real number x.

Let $h(x) = f(x) - g(x)$. Then $h(x) = 0$ for every rational number x.
Archimedes principle asserts that the rational numbers are dense in the reals.
Then for every real number z there is a sequence r_n of rationals such that $\lim_n r_n \to z$
Since $h(r_n) = 0$ for every n, and h is continuous on R, it follows that $h(z) = 0\ \forall z \in R$.
Then $f(x) = g(x)$ for every real number x.
Alternatively, suppose there is a real number z where $h(z) > 0$.
Then persistence of sign implies that there exists an $\varepsilon > 0$ such that $h(x) > 0\ \forall x \in N_\varepsilon(z)$
But since the rational numbers are dense in the reals $\forall \varepsilon > 0,\ \mathring{N}_\varepsilon(z) \cap \mathbb{Q} \neq \emptyset$
This contradiction implies there is no real number z where $h(z) > 0$.
Similarly, there can be no real number z where $h(z) < 0$.

2. Suppose that f is integrable on $I = [a,b]$. Let $\{P_n\}$ denote a sequence of partitions in $\Pi[a,b]$ such that for each n, P_{n+1} is a refinement of P_n. Explain why $s_n = s[f,P_n]$, the sequence of lower sums is a monotone increasing sequence that must converge. Explain why $S_n = S[f,P_n]$, the sequence of upper sums is a monotone decreasing sequence that must converge. Show that both sequences converge to the same limit.

It is a property of lower sums that if P_{n+1} is a refinement of P_n,
then $s_n = s[f,P_n] \leq s[f,P_{n+1}] = s_{n+1}$.
Then $\{s_n\}$ the sequence of lower sums is a monotone increasing sequence.
Since every lower sum is less than every upper sum, $\{s_n\}$ is bounded above by S_1.
The monotone sequence theorem asserts that this sequence converges to a limit, σ_*.
It is a property of upper sums that if P_{n+1} is a refinement of P_n,
then $S_n = S[f,P_n] \geq S[f,P_{n+1}] = S_{n+1}$.
Then $\{S_n\}$ the sequence of upper sums is a monotone decreasing sequence.
Since every lower sum is less than every upper sum, $\{S_n\}$ is bounded below by s_1.
The monotone sequence theorem asserts that this sequence converges to a limit, σ^*.
The definition of integrability for f states that if f is integrable on $I = [a,b]$, then $\sigma^* = \sigma_*$.

3. State the definition for a Cauchy sequence. Use the definition to prove that $a_n = \dfrac{2}{n^2+2}$ is a Cauchy sequence.

$\{a_n\}$ is a C-sequence if $\forall \varepsilon > 0,\ \exists M > 0$ such that $|a_n - a_m| < \varepsilon$, if $m > n > M$.
Suppose $m > n$. Then

$$
\begin{aligned}
|a_n - a_m| &= \left| \frac{2}{n^2+2} - \frac{2}{m^2+2} \right| \\
&= \left| \frac{2(m^2+2) - 2(n^2+2)}{(n^2+2)(m^2+2)} \right| \\
&= \frac{2}{(m^2+2)} \frac{m^2-n^2}{(n^2+2)} \\
&\le \frac{2}{m^2}\frac{m^2}{n^2} = \frac{2}{n^2} < \varepsilon \quad if \quad m > n > \sqrt{\frac{2}{\varepsilon}}
\end{aligned}
$$

4. State the mean value theorem for derivatives. Use this theorem to prove that if f is differentiable on its domain and if $|f'(x)| \le M$ for all x in the domain, then f is uniformly continuous on the domain. To which of the following functions does this argument apply, $f(x) = x^2$ on $[1,\infty)$ or $f(x) = 1/x^2$ on $[1,\infty)$?

Suppose $f(x) \in C[a,b] \cap C^1(a,b)$. Then there exists a point, $c \in (a,b)$, where

$$f(b) - f(a) = f'(c)(b-a)$$

If f is differentiable on its domain then for any $x > y$ in the domain,

$$f(x) - f(y) = f'(c)(x-y) \quad for \;\; c \;\; between \;\; x,y$$

If $|f'(x)| \le M$ for all x in the domain, then

$$|f(x) - f(y)| \; \le M|x-y|$$

It follows that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ if } |x-y| < \delta = \frac{\varepsilon}{M}$$

This is the definition of uniform continuity for f on its domain.
$f(x) = x^2$ is not uniformly continuous on $[1,\infty)$ since $f'(x) = 2x$ is not bounded on $[1,\infty)$
$f(x) = 1/x^2$ is uniformly continuous on $[1,\infty)$ since $f'(x) = 1/x \le 1$ on $[1,\infty)$.

5. State the mean value theorem for integrals. Use this theorem to prove that if f is continuous on the interval $[a,b]$, then there is a point c in $[a,b]$ where $f(c)$ equals the average value of f on $[a,b]$. Under what conditions on f will this point be unique?

Suppose $f(x)$ is continuous on $I = [a,b]$. Then there exists a point $c \in I$ such that

$$\int_I f = f(c)(b-a)$$

The average value of f on $[a,b]$ is defined as

$$\bar{f} = \frac{\int_a^b f}{b-a}$$

Then $\bar{f} = f(c)$. If f is monotone on $[a,b]$ then the point c is unique.

6. State the definition for the function $f(x)$ to be differentiable at an interior point c in its domain. Use the definition to compute the derivative of

$$f(x) = \int_0^{e^x} (e^{-t} + 1)dt$$

$f(x)$ is differentiable at an interior point c in its domain if

$$\lim_{h\to 0} D_h f[c] \quad \textit{exists.}$$

Then,

$$\begin{aligned} D_h f[c] &= \frac{\int_0^{e^{c+h}} (e^{-t} + 1)dt - \int_0^{e^c} (e^{-t} + 1)dt}{h} \\ &= \frac{\int_{e^c}^{e^{c+h}} (e^{-t} + 1)dt}{h} \\ &= \frac{(e^{-p} + 1)(e^{c+h} - e^c)}{h} \quad \text{for} \quad c < p < c + h \\ &= e^c(e^{-p} + 1)\frac{e^h - 1}{h} \end{aligned}$$

and since e^x is differentiable for all x, $\lim_{h\to 0} D_h f[c] = e^c(e^{-c} + 1) = 1 + e^c$

Here we used the additive property of the integral in step 2 and the MVT for integrals in step 3